Dynamic Optimal Control of Power Systems

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Outline

- Transients
- Generator swing equations
- Inverters
- Dynamic Load Models

3 Bus Network – Economic Dispatch

- Bus 1 load: 50 MW
- Bus 3 load: 75 MW
- Generator 1: Capacity 100 MW, Cost \$8/MW
- Generator 2: Capacity 40 MW, Cost \$2/MW

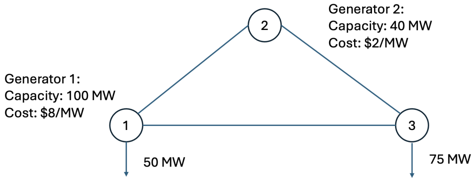


Figure: 3-Bus Power System Network

Quadratic Program (QP) formulation of ED

$$\min_{p_g} \quad \sum_{g \in \mathcal{G}} C_g(p_g) \tag{1}$$

$$\text{s.t.} \quad \sum_{g \in \mathcal{G}} p_g = \sum_{d \in \mathcal{D}} P_d \quad \text{(power balance)} \tag{2}$$

$$p_g^{\min} \leq p_g \leq p_g^{\max} \quad \forall g \in \mathcal{G} \quad \text{(capacity bounds)} \tag{3}$$

where:

- p_g: power output of generator g
- $C_g(p_g)$: cost function of generator g (often quadratic: $a_g p_g^2 + b_g p_g + c_g$)
- P_d: power demand at load d
- $\mathcal{G}$: set of generators, $\mathcal{D}$: set of loads

Exercise: Formulate the ED problem for the 3-bus network

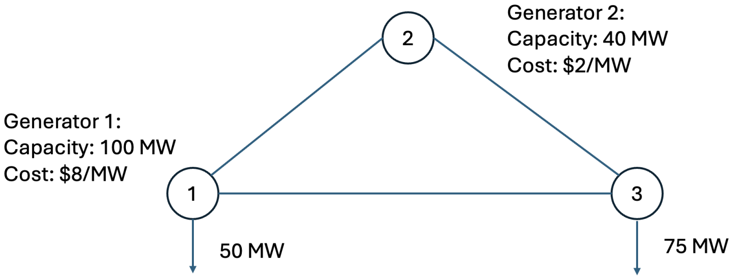


Figure: 3-Bus Power System Network

ED Formulation Answer

$$
\begin{aligned}
\min_{p_1, p_2} \quad & 8p_1 + 2p_2 && (4) \\
\text{s.t.} \quad & p_1 + p_2 = 125 \quad \text{(power balance)} && (5) \\
& 0 \leq p_1 \leq 100 \quad \text{(Gen 1 limits)} && (6) \\
& 0 \leq p_2 \leq 40 \quad \text{(Gen 2 limits)} && (7)
\end{aligned}
$$

Solution: $p_1 = 85$ MW, $p_2 = 40$ MW

- Total cost: $8 \times 85 + 2 \times 40 = 760$ \$/hour
- Gen 2 at maximum capacity (greedy)
- Gen 1 supplies remaining demand

Discussion Questions

What do you observe from your formulation?

- What kind of problem is this (linear, quadratic, etc.)?
- The power network is a graph – what type? What is missing here?
- The flow is not controllable - we did not place branch constraints.

What's the Problem?

- The graph should be directed: power has flow directions
- Line ratings and safety are ignored in ED
- Overloading lines is dangerous (thermal expansion, sag, wildfire risk)
- What is a power line:
  - Metal coil that expands and heats up when current is higher.
  - That's why we have rating (magnitude of power flow cannot exceed this amount). Physically you can exceed it (nothing is preventing the power to flow) a bit, but there are consequences above ...
- We need branch (line) constraints to ensure safe operation

DC Power Flow

Data:

- Generator set $\mathcal{G}_i$ at bus i (nodal generation)
- Load set $\mathcal{L}_i$ at bus i (nodal load)
- Costs $C_j(P_j)$ quadratic or piecewise-linear for generator j
- Line limits $F_\ell^{\max}$, generator bounds $P_j^{\min}, P_j^{\max}$

Decision variables:

- Generator outputs P_j for $j \in \mathcal{G}_i$
- Bus angles θ_i for $i \in \mathcal{N}$
- Line flows f_ℓ for $\ell \in \mathcal{L}$

DC Power Flow Formulation

$$\min_{P_j,\theta} \quad \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{G}_i} C_j(P_j) \tag{8}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{G}_i} P_j - \sum_{j\in\mathcal{L}_i} P_j = \sum_{k:(i,k)\in\mathcal{L}} \frac{1}{x_{ik}}(\theta_i - \theta_k) \quad \forall i \in \mathcal{N} \tag{9}$$

$$f_\ell = \frac{1}{x_\ell}(\theta_{i(\ell)} - \theta_{j(\ell)}), \quad -F_\ell^{\max} \le f_\ell \le F_\ell^{\max} \quad \forall \ell \in \mathcal{L} \tag{10}$$

$$P_j^{\min} \le P_j \le P_j^{\max} \quad \forall j \in \mathcal{G}_i, \forall i \in \mathcal{N} \tag{11}$$

$$\theta_{\text{ref}} = 0 \tag{12}$$

- x_{ij}: reactance of line. $1/x_{ij} = b_{ij}$: susceptance (manufacturer specified)
- Reference bus: only for modeling, you can pick any bus as the reference bus. We only care about angle differences (which carries current through lines
- Individual bus angle has no physical meaning

Exercise: Solve DCOPF (solver suggested: Ipopt)

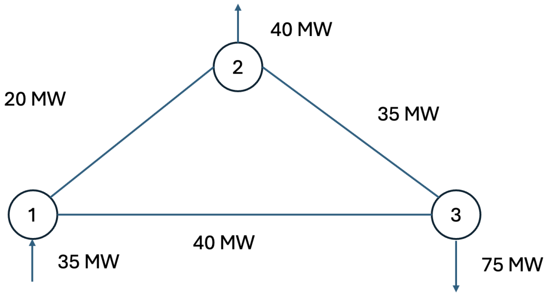
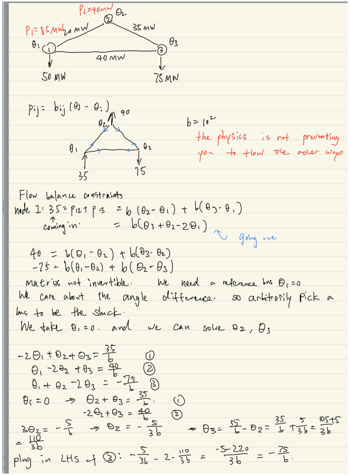


Figure: 3-Bus Network with Constraints

How did I get the numbers:

- Assume P1 generates 85 MW, with 50 MW of load, the net injection is 35 MW
- Assume P2 generates 40 MW, with no load, net injection is 40 MW (we take upwards arrow as injection)

DCOPF Solution



$p_{ij} = b_{ij}(\theta_j - \theta_i)$

$b = 10^2$

the physics is not preventing you to flow the other way

Flow balance constraints

node 1: $35 = p_{12} + p_{13} = b(\theta_2 - \theta_1) + b(\theta_3 - \theta_1)$

coming in $\quad = b(\theta_3 + \theta_2 - 2\theta_1)$ going out

$40 = b(\theta_1 - \theta_2) + b(\theta_3 - \theta_2)$

$-75 = b(\theta_1 - \theta_3) + b(\theta_2 - \theta_3)$

Matrix not invertible. We need a reference bus $\theta_1 = 0$

We care about the angle difference. so arbitrarily pick a bus to be the slack.

We take $\theta_1 = 0$. and we can solve θ_2, θ_3

$$-2\theta_1 + \theta_2 + \theta_3 = \frac{35}{b} \quad (1)$$
$$\theta_1 - 2\theta_2 + \theta_3 = \frac{40}{b} \quad (2)$$
$$\theta_1 + \theta_2 - 2\theta_3 = -\frac{75}{b} \quad (3)$$

$$\theta_1 = 0 \rightarrow \theta_2 + \theta_3 = \frac{35}{b}. \quad (1)$$
$$-2\theta_2 + \theta_3 = \frac{40}{b} \quad (2)$$

$$3\theta_2 = -\frac{5}{b} \rightarrow \theta_2 = -\frac{5}{3b} \rightarrow \theta_3 = \frac{35}{b} - \theta_2 = \frac{35}{b} + \frac{5}{3b} = \frac{105+5}{3b} = \frac{110}{3b}$$

plug in LHS of (3): $-\frac{5}{3b} - 2 \cdot \frac{110}{3b} = \frac{-5-220}{3b} = -\frac{75}{b}$.

Figure: DCOPF Solution Results

DCOPF Solution Details

$p_{12} = b\left(-\frac{5}{36} - 0\right) = -\frac{5}{3}$

$p_{13} = \frac{110}{3}$

$p_{23} = \frac{115}{3}$

Not over determined. not full rank   2-3 is overloaded

Reformulate using branch constraints

DCOPF: although not controllable, we take θ as dec. vars.

$$\min \quad 8p_1 + 2p_2$$

$$\text{s.t.} \quad p_1 \le 100, \quad p_2 \le 40$$

$$p_{1,2} \ge 0$$

power @ 2 : $p_2 = b(\theta_3 + \theta_1 - 2\theta_2)$

$$p_1 - 50 = b(\theta_2 + \theta_3 - 2\theta_1)$$

$$-75 = b(\theta_1 + \theta_2 - 2\theta_3)$$

Solver will pick a reference bus

Figure: DCOPF Detailed Analysis

Wrap Up

- You will see that without thermal limits, optimal dispatch can overload lines
- Reference bus is arbitrarily picked by the solver.
- Real systems are AC (complex voltages/currents) – much harder. This is just a lightweight intro so we can think about expressing real-world problems as optimization formulations without overburdening ourselves with AC physics, which we will see in transient stability section.

Transient Dynamics

What are transients?

When current or voltage changes suddenly — switching, faults, lightning, equipment failures, etc. — the system experiences a **transient**

- Transients are short-lived, high-frequency events where stored magnetic and electric energy exchange rapidly.
- **Faraday's law** of electromagnetic induction governs these effects:

  A change in magnetic flux through a circuit induces a voltage across it.

  $$v(t) = \frac{d\Phi(t)}{dt}$$

  where $\Phi(t)$ is the magnetic flux through the circuit.

Transients Continued

- For an inductor, the magnetic flux Φ is proportional to the current:

$$\Phi(t) = L\, i(t)$$

  where L is the inductance (magnetic energy stored per unit current).
- Substituting gives the familiar time-domain voltage rule:

$$\boxed{v_L(t) = L\,\frac{di(t)}{dt}}$$

Note that steady-state phasor analysis no longer holds due to the time-varying nature of the magnetic flux. I will draw the connection later.

Sinusoidal steady state

Assume all quantities have angular frequency ω:

$$i(t) = \mathrm{Re}\left\{ I \, e^{j\omega t} \right\},$$

Differentiate the current:

$$\frac{di(t)}{dt} = \mathrm{Re}\left\{ j\omega I \, e^{j\omega t} \right\}.$$

Substitute into $v_L(t) = L \, \frac{di}{dt}$:

$$v_L(t) = \mathrm{Re}\left\{ (j\omega L I) \, e^{j\omega t} \right\}.$$

Phasor (frequency-domain) relation

By definition, the **phasor** is the complex amplitude multiplying $e^{j\omega t}$.
From the previous expression,

$$v_L(t) = \mathrm{Re}\left\{(j\omega L I)\, e^{j\omega t}\right\},$$

so the **voltage phasor** is

$$\boxed{V = j\omega L\, I}.$$

Capacitor law: from time domain to phasor domain

Physical basis:

- A capacitor stores energy in an **electric field**.The stored charge $q(t)$ is proportional to voltage:

$$q(t) = C\, v(t)$$

  where C is the capacitance.

- The current is the rate of change of charge:

$$\boxed{i_C(t) = \frac{dq(t)}{dt} = C\,\frac{dv(t)}{dt}}.$$

Under sinusoidal steady state:

$$v(t) = \mathrm{Re}\left\{ V e^{j\omega t} \right\}$$

where V is the voltage phasor.

Capacitor law: from time domain to phasor domain

$$\frac{dv(t)}{dt} = \mathrm{Re}\left\{j\omega V e^{j\omega t}\right\},$$
$$i_C(t) = \mathrm{Re}\left\{(j\omega C V) e^{j\omega t}\right\}.$$

Hence, the **phasor relationship** is:

$$\boxed{I = j\omega C\, V}, \qquad \boxed{Y_C = j\omega C}, \qquad \boxed{Z_C = \frac{1}{j\omega C}}.$$

You could of course derive admittance and impedance for inductors following similar steps. This is how you go from time domain to phasor domain. Note that the above is for ideal inductors and capacitors.

More realistic transmission line model

The voltage $v(x,t)$ and current $i(x,t)$ vary **both** in time and along the line coordinate x. Their spatial derivatives represent how these quantities change **per unit length:**

$$\frac{\partial v(x,t)}{\partial x} \Rightarrow \text{voltage drop per unit length (V/m)},$$
$$\frac{\partial i(x,t)}{\partial x} \Rightarrow \text{current change per unit length (A/m)}.$$

Real lines are lossy:

- Conductor series resistance causes Ohmic losses (heat dissipation) $\Rightarrow$ add $-R'\,i(x,t)$.
- Current leakage due to shunt conductance $\Rightarrow$ add $-G'\,v(x,t)$.

Hence, the full **telegrapher's equations** become:

More on realistic transmission line model

$$\frac{\partial v(x,t)}{\partial x} = -L'\frac{\partial i(x,t)}{\partial t} - R'\, i(x,t),$$
$$\frac{\partial i(x,t)}{\partial x} = -C'\frac{\partial v(x,t)}{\partial t} - G'\, v(x,t).$$

You can think about R' and G' as damping terms. L and C relate to energy storage, and R and G relate to energy dissipation.

How the above was derived

Setup: Consider a small transmission line segment between x and $x + dx$.

- x increases in the direction of current flow $(+x)$.
- $i(x, t)$: current flowing in $+x$ direction.
- $v(x, t)$: voltage between conductors (top to bottom) at position x.

1. Voltage change between segment ends:

$$v(x, t) - v(x + dx, t) = -\frac{\partial v(x, t)}{\partial x}\, dx.$$

2. Series drops over dx:

$$\begin{aligned} \text{Resistive drop}: &\ \ R'\, i(x, t)\, dx, \\ \text{Inductive drop}: &\ \ L' \frac{\partial i(x, t)}{\partial t}\, dx. \end{aligned}$$

How the above was derived

3. Apply Kichhoff Voltage Law:

(The sum of voltage drops along the closed path must equal zero.)

$$(R'\, i + L' \tfrac{\partial i}{\partial t})\, dx + \big[v(x + dx, t) - v(x, t)\big] = 0.$$

4. Substitute and simplify:

$$\frac{\partial v(x,t)}{\partial x}\, dx = -L' \frac{\partial i(x,t)}{\partial t}\, dx - R'\, i(x,t)\, dx.$$

5. Divide by dx:

$$\boxed{\frac{\partial v(x,t)}{\partial x} = -L' \frac{\partial i(x,t)}{\partial t} - R' i(x,t)}.$$

The negative sign indicates that voltage *drops* in the $+x$ direction due to both inductive ($L'\, \partial i/\partial t$) and resistive ($R' i$) effects.

How does physics relate to optimization?

Transient Stability Constrained Optimal Power Flow (TSCOPF)

TSCOPF formulation

$$
\begin{aligned}
\min_{p,\,x(t),\,y(t)} \quad & C(p) && (1)\\
\text{s.t.} \quad & g_s(p) = 0 && (2)\\
& h_s^- \le h_s(p) \le h_s^+ && (3)\\
& p^- \le p \le p^+ && (4)\\
& \dot{x} = f(x, y, p), \quad x(t_0) = I_x^0(p) && (5)\\
& 0 = g(x, y, p), \quad y(t_0) = I_y^0(p) && (6)\\
& h(x(t), y(t)) \le 0, \quad \forall t && (7)
\end{aligned}
$$

Objective: minimize operating cost or transmission losses. (2) includes steady-state nodal power balance constraints. (3) includes apparent/real power/reactive power/current flow constraints on lines. (4) includes generator capacity or voltage magnitude constraints.

Dynamic Transient Constraints: (5)–(7)

Eq. (5):

- x: state variables (rotor angles, speeds, control states). Initial states computed from steady-state solution corresponding to control variables p.
- $f(x, y, p)$: system dynamics — e.g., generator swing equations, Telegrapher equations, or capacitor/inductor transient models.
- y: dependent variables (nodal voltages magnitude and angle, line currents, etc.).
- p: control variables (generator setpoints, tap settings, shunt positions, etc.)
- Enforce the physics of transient after a disturbance.

Eq. (6): embed dynamics into steady-state constraints.

- $g(x, y, p)$: Same physical laws as (2) e.g. KCL but now applied at every instant t's states $x(t), y(t)$ to extend to the dynamics.

Dynamic and Transient Constraints: (7)

(7) Transient limits:

$$h(x(t), y(t)) \leq 0, \quad \forall t$$

- Enforce time-domain operating limits during the transient response.
- Examples:
  - Bus voltage magnitudes stay within limits.
  - Rotor angle differences remain stable.
  - Line thermal limits respected.
- Ensures **transient stability** under all time steps during instability.

Solution Methods for TSC-OPF

Indirect (variational) Methods:

- Based on Pontryagin's Maximum Principle.
- Replace the differential equations of dynamics with inequalities that approximate the behavior in steady-state by linearizing into static conditions.
- Examples: energy or Lyapunov functions or impose stability margin constraints on linearized Jacobian.

Instead of having to integrate over time, you get back a static nonlinear optimization problem that can be solved using standard solvers. **In practice:**

- Mainly used for planning/screening/preventive security dispatch due to loss in accuracy.
- Not sufficient to guarantee transient stability under large disturbances.
- Validation still relies on time-domain (direct) simulation.

Direct Method: Simultaneous Discretization/Constraint Transcription

Main idea: Converts the time-dependent diff. eq. into a finite set of algebraic constraints before optimization so transient stabilitysimulator can be reused.

Discretization approach:

- The simulation horizon is divided into multiple time steps $t_0, t_1, \ldots, t_N$.
- The diff. eq. is approximated at each step using numerical integration like implicit trapezoidal rule:

$$x(t) - \frac{\Delta t}{2} f(x(t), y(t), p) - x(t - \Delta t) - \frac{\Delta t}{2} f(x(t - \Delta t), y(t - \Delta t), p) = 0.$$

Pros and Cons:

- Produces one large-scale NLP that enforces the dynamics exactly for the entire trajectory (within discretization accuracy).
- Computationally demanding due to the high dimensionality of variables and constraints from discretization. Accurate gradients is expensive from trajectory sensitivity analysis. Hessians often approximated using BFGS updates.

Direct Method: Multiple Shooting

The multiple shooting method divides the simulation horizon into smaller time segments $[t_0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N]$.

- Each segment starts from its own initial condition $x_i(t_i)$ and is integrated forward using the diff. eq. $\dot{x} = f(x, y, p)$, $0 = g(x, y, p)$ to obtain the predicted final state $\hat{x}_i(t_{i+1})$.
- Constraint to ensure continuity between segments:

$$x_{i+1}(t_{i+1}) = \hat{x}_i(t_{i+1}),$$

Constraint form:

$$s_i = S_i(s_{i-1}, p), \quad \forall i \in 1, \ldots, N_S,$$

where $S_i(\cdot)$ is an implicit function that can be numerically integrated over segment i.
Pros: Each segment can be integrated independently, so the Jacobian of the resulting NLP is better conditioned because the coupling is limited to adjacent segments instead of the entire trajectory. This segmentation improves numerical stability and allows for more efficient large-scale computation.

Trajectory Sensitivity Analysis of TSC-OPF

Purpose: Quantify how system variables $x(t), y(t)$ changes with respect to small variations in control variables p or intial conditions. Recall that with different control settings p, the entire transient trajectory changes and we would need to simulate the dynamics again to see what happens. This is expensive. Sensitivity analysis tells you how the trajectory and stability margins change with small changes in $p : \frac{\partial x}{\partial p}$ without running a new full simulation for every small perturbation.

Relation to numerical methods:

- These sensitivities provide gradient information for solvers, which is used for both multiple shooting and constraint transcription.

Forward Sensitivity Method

- Computed by performing a forward integration of the sensitivity equations alongside the original diff. eq. system.
- Efficient when the number of parameters is small.
- The computational complexity is $\mathcal{O}(n_p)$, since n_p forward integrations are required to compute the sensitivities.

Formulation: For the original diff. eq. system:

$$F(\dot{x}(t), x(t), p) = 0.$$

The corresponding variational diff. eq. for the sensitivities is:

$$\frac{\partial F}{\partial \dot{x}} \dot{s} + \frac{\partial F}{\partial x} s + \frac{\partial F}{\partial p} = 0,$$

with initial condition

$$s(t_0) = \frac{\partial x(t_0)}{\partial p}.$$

Forward Sensitivity Method

- Each parameter p_i perturbs the system differently.
- Forward method tracks this by integrating a new "copy" of the linearized system, which shares the same Jacobian as the original DAE.

Pros and cons:

- **Pros:** Simple and accurate, efficient when number of parameters is small.
- **Cons:** Computational cost grows linearly with number of parameters.

Adjoint Method

Efficient when the number of parameters is large. This only needs one backward integration in time to compute the sensitivities. Formulation:

- Consider

$$G(p) = \int_{t_0}^{T} g(x, p)\, dt.$$

- We want the gradient given by:

$$\frac{\partial G}{\partial p} = \int_{t_0}^{T} \left(\frac{\partial g}{\partial p} - \lambda^{\mathsf{T}} \frac{\partial F}{\partial p} \right) dt,$$

- The adjoint multiplier $\lambda(t)$ satisfies

$$\dot{\lambda} = -\frac{\partial g}{\partial x} + \lambda^{\mathsf{T}} \frac{\partial F}{\partial x}.$$

where $\lambda(T) = 0$.

Adjoint Method and wrapping up

Pros:

- Efficient when the number of parameters is large.
- The gradient is obtained in one pass.

Cons:

- Higher memory cost due to storage of trajectory data and state variables in backward integration.

One can also obtain the gradients by finite differences, which is based on truncated Taylor series expansion.

Power System History and Modern Power System

The Fuel Era (20th Century)

Electricity produced mostly by coal, gas, nuclear. Generators are large synchronous machines with big spinning masses. Stable and predictable. Inertia from these machines naturally provides flexibility infrequency stability. Grid ran reliably for decades.

The Renewable Era (2000s–Today)

Wind expanded in 2000s, solar PV took off after 2010. Renewables now 20–40%+ of real-time demand in some regions; dynamics changed.

Synchronous Generators: How electricity is generated

- Rotor (heavy spinning mass) driven by turbines (steam, gas, hydro)
- Faraday's law: changing magnetic field induces voltage in stator
- Called "synchronous" because the rotor spins in sync with the grid's frequency (50 Hz in Europe, 60 Hz in North America)
- If the grid frequency is 60 Hz, the rotor turns at a speed locked to 60 Hz

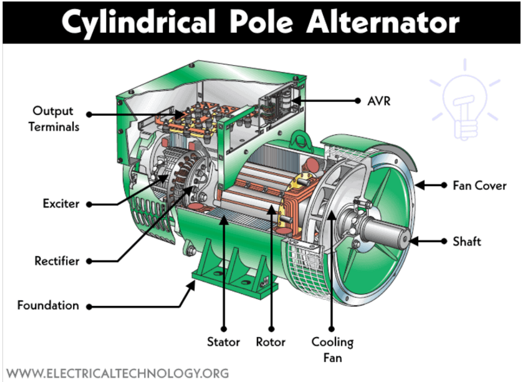


Figure: Generator Cross-Section

Spinning Mass in a Generator

- Inside a synchronous generator is a rotor — basically a giant heavy wheel of steel and copper (tens or hundreds of tons)
- Turbines (steam from coal/nuclear, gas combustion, or flowing water in hydro) push on the rotor to make it spin
- That rotor's mechanical rotation creates a rotating magnetic field, according to Faraday's law of induction, a changing magnetic field induces an alternating voltage in the stator windings
- This is why the system is predictable: we know how to control these rotors. Put in more fuel to generate more power

Generator Frequency Formula

Frequency Formula:

$$f = \frac{N \times \mathrm{RPM}}{120} \tag{13}$$

where N = number of poles, RPM = rotor speed

Examples:

- 2 poles, 3600 RPM $\rightarrow$ 60 Hz
- 4 poles, 1800 RPM $\rightarrow$ 60 Hz

Why 50/60 Hz? Historical choices: early engineers (Westinghouse, Edison, etc.) picked values that balanced motor performance and generator design. Once infrastructure was built, it became a standard.

Kinetic Energy

The rotor has stored kinetic energy:

$$E_{\text{kinetic}} = \frac{1}{2} J\omega^2 \tag{14}$$

where J = moment of inertia (depends on mass + geometry), ω = rotor speed

If demand suddenly exceeds supply (a generator trips):

- That small slow down of a rotor releases some of its stored kinetic energy into the grid instantly
- But because there are so many large spinning machines, the grid behaves like a conveyor belt with so many wheels tied together. If one slows a bit, the others share the imbalance, so frequency changes slowly because the system has a huge inertia
- This gives time for operators to fix things
- Even if there are imbalances, things wouldn't get out of hand fast since there are so many other generators. They can share the load so each only needs to spin a little faster to keep up the frequency

Inverters - Renewables

Today, renewables can supply 20–40%+ of real-time demand. Cleaner, cheaper, more sustainable — but dynamics changed.

Most renewables (solar PV, modern wind turbines, batteries) produce DC electricity (direct current).

What's the problem of DC power?

- It only has amplitude (magnitude of voltage/current)
- No phase, no frequency
- But recall AC current has the waveform (that's why we have leading/lagging current which controls reactive power and power factor correction)
- We need amplitude, frequency, and phase to describe AC current
- That's why we need inverters, power electronics device that synthesizes sinusoidal AC from DC

Inverter Operation

How it operates?

1. Takes DC input from solar panels, wind turbine
2. Use power electronics that switches thousands of time per second to synthesize an AC waveform
3. Note that even the output is a smooth sinusoidal AC waveform, inside the inverter the switches turn the DC voltage on and off thousands of times per second (typical switching frequency = 2–20 kHz, sometimes higher) to approximate that smooth waveform
4. So even though the output is continuous, it's created by on/off pulses internally
5. The inverter synchronizes the AC output to the grid's frequency and phase. If grid is 60 Hz $\rightarrow$ inverter outputs 60 Hz. If grid is 59.9 Hz (after a disturbance) $\rightarrow$ inverter follows 59.9 Hz.
6. The voltage, current, and power factor are controlled through the programmed algorithms

Inverter Control Modes

In summary, the inverters are programmable devices by operators with control algorithms to act like generators. They wait for a signal from a grid so they can be:

- **Grid-following**: track the grid's voltage and frequency $\rightarrow$ inject current accordingly
- **Grid-forming**: behave like a voltage source, set their own frequency/voltage reference, and to adjust for power imbalance (some research area I heard of)

They are not really generators - no spinning mass, no inertia, but they use control algorithms to mimic generator behavior.

Internal View of Inverters

Figure: Internal View of Inverter

- Capacitors and switching components on electronic mainboards (like in computer's motherboard, blue cylinders in upper left corner of the picture)
- Programmable behavior defined by control firmware

Problems with Inverters

This is all software-based. You do not have a natural physical property like a spinning rotor and inertia.

- No big spinning mass directly tied to frequency, so frequency changes much faster after a disturbance
- The device measures grid signal and forces its output to follow
- Unless explicitly programmed, they don't know when the conveyor belt slows down or speeds up (recall the previous analogy)
- Even if they do, they don't have the capacity like big generators
- **This is the key part:** they are just switching circuits with no agency to ramp up the power output (nature of renewables is their output is often independent of human control). Output is limited by weather and energy availability (sun/wind).
- Renewables also locate in remote areas with long transmission lines, and the nature of their unpredictability (weather), makes their generation highly uncertain

We will build up to inverter control after we cover the generator swing equations.

Newton's Second Law

Linear Version:

$$F = ma \tag{15}$$

where F = force (N), m = mass (kg), a = acceleration (m/s²)
This says: imbalance of forces $\rightarrow$ acceleration of mass.
Rotational Version: For a rotating body (like a generator rotor), the equation is:

$$T = J\alpha \tag{16}$$

where T = torque (N·m), J = moment of inertia (kg·m²), the rotational mass. α = angular acceleration (rad/s²)
This says: imbalance of torques $\rightarrow$ rotor accelerates or decelerates. Think torque as the angular equivalent of force.

Applied to Generator Dynamics

Two main torques act on a synchronous generator's rotor:

- T_m: mechanical torque from the turbine (steam, gas, water) pushing the rotor
- T_e: electromagnetic torque from the stator's magnetic field resisting the rotor (this is the grid "pulling" power out)

Torque imbalance:

$$J\alpha = T_m - T_e \tag{17}$$

where ω: angular speed of rotor (rad/s), $\alpha = \dot{\omega}$: angular acceleration (rad/s²)

If $T_m > T_e$: rotor accelerates

If $T_m < T_e$: rotor slows down

If equal: steady rotation

From Torque to Power

Recall $P = Fv$ (mechanical power is generated by a force F on a body moving at a velocity v). In rotational systems, power is related by torque and angular speed (you can think about it as rotational equivalent as force)

Power = torque × speed:

$$P = T \cdot \omega \tag{18}$$

So:

- Mechanical power input: $P_m = T_m \cdot \omega$
- Electrical power output: $P_e = T_e \cdot \omega$

Multiply the torque balance by ω:

$$J\omega\dot{\omega} = P_m - P_e \tag{19}$$

This relates how fast the mass is spinning (ω) to the imbalance of power input (generation) and power withdrawal (load + losses).

From Torque to Power (Continued)

But recall that generators operate close to system frequency, so the generators spin at, i.e. the angular velocity is close to that 60 Hz constant. Since the variations are mostly tiny, we can define inertia constant $M = J\omega$
And we get the generator swing equation:

$$M\dot{\omega} = P_m - P_e \tag{20}$$

Interpretation:

- M: inertia constant, measures how much the rotor resists speed change (bigger mass $\rightarrow$ slower frequency drift)
- P_m: mechanical input power (from fuel, water, steam)
- P_e: electrical output power delivered to the grid

Per-Unit Generator Swing Equation

Per-unit versions: Power are often defined at per unit, so we have:

$$\frac{J\omega_s}{S_{\text{base}}}\dot{\omega} = (P_m - P_e)_{\text{pu}} \tag{21}$$

There are sources that define $H = \frac{1}{2}\frac{J\omega_s^2}{S_{\text{base}}} \Rightarrow \frac{2H}{\omega_s} = \frac{J\omega_s}{S_{\text{base}}}$
$H \triangleq \frac{E_k}{S_{\text{base}}} = \frac{\frac{1}{2}J\omega_s^2}{S_{\text{base}}}$ comes from kinetic energy

So we have per unit swing:

$$\frac{2H}{\omega_s}\dot{\omega} = (P_m - P_e)_{\text{pu}} \tag{22}$$

Damping and Advanced Forms

Some also add damping:

$$M\dot{\omega} + D(\omega - \omega_s) = P_m - P_e \tag{23}$$

as penalties to frequency deviation. D captures any restoring force, or frictions and losses
We can also write the per-unit acceleration form:

$$2H\dot{\omega}_{\text{pu}} = (P_m - P_e)_{\text{pu}} \tag{24}$$

Why This Matters

- Stability depends on balancing generation and demand
- Inertia slows down changes in frequency, buying time for control actions

Power imbalance effects:

- If $P_m > P_e$: extra power $\rightarrow$ rotor speeds up $\rightarrow$ frequency rises
- If $P_m < P_e$: shortage $\rightarrow$ rotor slows $\rightarrow$ frequency falls

Inertia M slows down how fast this happens.

- With many generators, M is big. For a given imbalance, frequency drifts slowly $\rightarrow$ grid is flexible
- With fewer machines (more renewables), M is smaller $\rightarrow$ frequency changes faster $\rightarrow$ grid is fragile

This is shown by writing the equation as $\dot{\omega} = \frac{P_m - P_e}{M}$. For a big M, the angular acceleration (frequency) is smaller when there is an imbalance.

How Does It Relate to Inverters?

We previously discussed grid-following inverters.

Its control law works as:

1. Measure voltage and frequency of the grid
2. Gets desired power output from an operator or solved from a market
3. Adjust its AC current output to deliver the desired power output at voltage and frequency GIVEN by the grid

What's the problem with it? No control over other parameters like voltage and frequency. The power output is only stable if the rest of the grid is stable.

Grid-forming Inverters

- The inverter doesn't blindly follow the grid frequency, it defines its own reference voltage and frequency like a voltage source
- Let the frequency shift slightly to reflect power imbalance between the renewable generation and the rest of the grid, so other machines (generators) know to ramp up or down
- Able to define its own frequency is the key for the renewable to behave like a synchronous generator, and we now have the ability to model it in a swing equation by giving it a virtual mass defined by the local frequency

We can now emulate synchronous machine behavior via controlling the "virtual inertia".

Reference: J. Driesen and K. Visscher, "Virtual synchronous generators," 2008 IEEE Power and Energy Society General Meeting - Conversion and Delivery of Electrical Energy in the 21st Century, Pittsburgh, PA, USA, 2008, pp. 1-3.

Virtual Inertia Law in Grid-Forming Inverter

Physics is replaced by software:

$$M_{\text{virtual}}\dot{\omega} = P_{\text{ref}} - P \tag{25}$$

Parameters:

- M_{virtual}: a tunable parameter by the controller, chosen to represent how fast the inverter responds to frequency changes
- Recall $M = J\omega$. Since we have no physical inertia J, the virtual mass is just a modeling choice
- P_{ref}: reference active power dispatch (from operator)
- P: measured actual active power delivered
- ω: inverter's internal frequency reference value

Effect of M_{virtual}:

- A larger M_{virtual} means the inverter allows its frequency to drift slower, a "heavier" machine
- A smaller M_{virtual} means the inverter reacts more quickly, a "lighter" machine

Virtual Inertia (continued)

How it works:

- The inverter adjusts its internal frequency reference according to power imbalance
- That frequency reference drives its voltage output, which the grid "sees"
- To the rest of the system, this looks just like a synchronous machine rotor slowing/speeding under imbalance
- There's no heavy rotor. It emulates the inertia behavior of synchronous machines through software control.
- But this is not sustained. It can only hold it until the renewable saturates, which is less than a second since they don't have as much buffer as traditional generators.
- Other problems: semiconductor ratings, thermal limits, hard to tune M_{virtual}, cost, legacy devices, etc.

Why this is such a big deal: renewables can now respond to a grid-wide drop in frequency because it can behave almost like a synchronous generator through control law

Grid-following inverters - Droop Control

Generators naturally slow down if overloaded, resulting in a drop in frequency (droop). Droop control allows each generator to increase its power output in response, but in proportion to its droop coefficient, so that all generators share the load change fairly.
Grid-forming inverters are programmed with droop control:

$$P = P_{\text{set}} - \frac{1}{K_p}(\omega - \omega_0) \tag{26}$$

where ω_0: nominal frequency, P_{set}: reference power output, K_p: droop constant (rad/s per MW or Hz per MW) telling inverter how much to adjust power output when frequency changes. Hence when frequency drops, power will rise.
Overload leads to frequency drop, and power will rise according to the relationship. If frequency rises, power will decrease to maintain the frequency.

Reactive Power and Voltage Control (Q-V Droop)

Analogous for voltage support:

$$V = V_0 - K_q(Q - Q_{\text{set}}) \tag{27}$$

or

$$Q = Q_{\text{set}} - \frac{1}{K_q}(V - V_0) \tag{28}$$

If reactive demand $\uparrow$ (voltage dips), generator/inverter increases reactive power injection.
Two important components of stability: frequency and voltage. If there are deviations, adjust active and reactive power accordingly

- $1/K_p$ = MW per Hz: "How much active power do I add if frequency drops by 0.1 Hz?"
- $1/K_q$ = Mvar per V $\rightarrow$ "How much reactive power do I add if voltage drops by 0.01 pu?"

Dynamic Load Models

Steady-State Load Models in ACOPF/DCOPF

In optimal power flow (OPF), all quantities are *time-invariant*. $\dot{x} = 0, t$ does not appear. At each load bus:

$$P_D = \text{fixed real power demand}, \qquad Q_D = \text{fixed reactive demand}.$$

These loads can be either:

- Constant power: P_D, Q_D are specified numerical values; or
- Voltage-dependent (ZIP) as motors draw different amount current to maintain torque:

$$P(V) = P_0(a_P V^2 + b_P V + c_P), \quad Q(V) = Q_0(a_Q V^2 + b_Q V + c_Q).$$

Power-flow balance:

$$P_G - P_D = \text{network losses}, \qquad Q_G - Q_D = 0.$$

Steady-State Load Models Continued

Interpretation of the above model:

- Loads are fixed regardless of system conditions, or at most respond to nodal voltage.
- No memory or dynamics – they change only between static operating points.
- The OPF represents a single equilibrium snapshot of the system.

Next: Dynamic models generalize this by letting P_D and Q_D *evolve over time* with voltage and frequency. The grid's sink can be dynamic too even though we tend to think about it as a fixed parameter.

Dynamic Load Models - Induction Motor Model

In dynamic models, the active and reactive power is represented as a function of the past and present voltage magnitude and frequency of the load bus. This type of model is commonly derived from the equivalent circuit of an induction motor.
Most real-world loads (fans, pumps, compressors) are induction motors. Their active/reactive power do not change instantaneously with voltage change but rather dynamically by changing the motor's rotor speed ω_r to maintain torque. Hence, the power consumption depends on both voltage and frequency.

$$P_d = f_P(V, \omega_r), \qquad Q_d = f_Q(V, \omega_r),$$

so the load has internal dynamics, unlike static P_D, Q_D in ACOPF.

Rotor Dynamics

Slip:

$$s = \frac{\omega_s - \omega_r}{\omega_s},$$

where

- ω_s: synchronous electrical speed ($2\pi f_s$), set by system frequency,
- ω_r: mechanical rotor speed (frequency at the load).

Operating regions:

- $s = 0$: rotor synchronous with system frequency $\rightarrow$ no induced torque.
- $0 < s < 1$: normal operation. Rotor is slightly slower than the rotating field $\rightarrow$ induces current $\rightarrow$ produces torque.
- $s \rightarrow 1$: stall. Rotor stopped, max current, high losses.

Mechanical dynamics (sign flip if you differentiate w.r.t. slip):

$$J\frac{d\omega_r}{dt} = T_e(V, \omega_r) - T_m \quad \Longleftrightarrow \quad J\omega_s \frac{ds}{dt} = T_m - T_e(V, s).$$

Continued

Variable definitions:

- J: rotor inertia (kg·m^2),
- ω_r: rotor mechanical speed/frequency (rad/s),
- T_e: electromagnetic torque (depends on bus voltage V, frequency ω_s (slip formulation), or ω_r (standard formulation)),
- T_m: mechanical load torque.

Example: Induction Motor Response During a Voltage Sag

Sequence of events:

1. **Voltage drop:** V suddenly decreases.
2. **Torque imbalance:** Electromagnetic torque $T_e(V, \omega_r)$ falls below mechanical torque T_m.
3. **Rotor slowdown:** ω_r decreases $\Rightarrow$ slip s increases.
4. **Increased current and VAR demand:** The motor draws more current to restore torque, which further depresses voltage.
5. **Possible stalling:** If V remains low, the motor stalls — reactive power skyrockets $\Rightarrow$ voltage collapse.

IM Modeling is key to capture this nonlinear instability mechanism.

How TSC-OPF Prevents Motor Stalling and Voltage Collapse

In TSC-OPF, dynamics and limits are enforced directly:

$$\dot{x} = f(x, y, p), \qquad 0 = g(x, y, p), \qquad h(x(t), y(t)) \leq 0, \ \forall t.$$

Dynamic states x:

- Induction motor slip s, generator rotor angles, inverter controls, etc.
- Their evolution $f(x, y, p)$ describes how voltages and speeds change after a disturbance.
- Dynamic constraints ensure stall condition is not reached.

Constraint function $h(x(t), y(t))$:

- Enforces time-domain limits such as

  $$V_i(t) \geq V_{\min}, \quad \forall t \in [0, T_{\text{rec}}],$$

  where T_{rec} is the recovery time. This ensures bus voltages remain within safe recovery bounds during the entire horizon after a disturbance.
- Prevents continued voltage sag that would drive $T_e(V, s)$ down and cause stalling.

Wrap up

Controls p are chosen so that f remains stable under disturbances, and the chain of events above is prevented since motor and network dynamics are embedded in f, g, h.
This model is typically used when there's a fast transient or stalling, and the time scale is in milliseconds to second. Useful for short-term voltage stability and transient studies.

Exponential Recovery Load (ERL): Motivation and Concept

Goal: Represent aggregate load behavior during voltage recovery after a disturbance.
Empirical Observation:

- When voltage dips, total active and reactive loads drop immediately.
- Loads such as motor controls and HVAC systems **slowly restore** their power draw as voltage recovers.
- The recovery follows an **exponential time pattern**, not an instantaneous jump.

Idea: Introduce internal states that describe this gradual return:

$$P_d(t) = f_P(V(t), x_p(t)), \qquad Q_d(t) = f_Q(V(t), x_q(t)).$$

where $x_p(t)$ and $x_q(t)$ are the internal states of the active and reactive load.
These states evolve according to first-order differential equations, capturing the "memory" of how far the load has recovered.

Adaptive Exponential Recovery Load (ERL) Model

Mathematical Form:

$$T_p \frac{dx_p}{dt} = -x_p \left(\frac{V}{V_0}\right)^{N_{ps}} + P_0 \left(\frac{V}{V_0}\right)^{N_{pt}},$$

$$P_d = x_p \left(\frac{V}{V_0}\right)^{N_{pt}},$$

$$T_q \frac{dx_q}{dt} = -x_q \left(\frac{V}{V_0}\right)^{N_{qs}} + Q_0 \left(\frac{V}{V_0}\right)^{N_{qt}},$$

$$Q_d = x_q \left(\frac{V}{V_0}\right)^{N_{qt}}.$$

Parameters:

- x_p, x_q: internal recovery states (how much of the load has recovered).
- T_p, T_q: time constants — larger values $\Rightarrow$ slower recovery.

Continued

- P_0, Q_0: nominal power withdrawals at reference voltage V_0.
- N_{pt}, N_{qt}: transient exponents (immediate voltage sensitivity). How sharply load power reacts immediately when voltage changes (the short-term dip).
- N_{ps}, N_{qs}: steady-state exponents (long-term voltage dependence). How much the load power changes in the long term after voltage settles to a new level.

Interpretation:

- After a voltage dip, power first follows the transient curve, then recovers exponentially toward the steady-state curve.
- When V is weak, the recovery term $(V/V_0)^{N_{ps}}$ slows the rate of change — recovery stalls under low voltage.
- Recovery speed and power response both scale with voltage and almost halts under deep voltage sag.

Standard vs. Adaptive ERL Models

Standard ERL model:

$$T_p \frac{dx_p}{dt} = -x_p + P_0 \left[\left(\frac{V}{V_0} \right)^{N_{ps}} - \left(\frac{V}{V_0} \right)^{N_{pt}} \right],$$

$$P_d = x_p + P_0 \left(\frac{V}{V_0} \right)^{N_{pt}}.$$

Key difference:

- In the **standard** model, dx_p/dt depends on V only through the voltage-dependent driving term. The state x_p recovers constantly—independent of voltage.
- In the **adaptive** model, recovery slows when V is low:the term $(V/V_0)^{N_{ps}}$ reduces the rate of change. The two differ in how strongly the voltage affects recovery speed.

Physical interpretation:

- Standard ERL: suitable for moderate voltage dips.
- Adaptive ERL: more realistic for deep voltage sags.

ERL Parameters

- Parameters (T_p, T_q, N_{ps}, N_{pt}, N_{qs}, N_{qt}) are fitted empirically from from measurements of load recovery after voltage disturbances.
- They are not linked to physical machine constants, but to observed aggregate behavior of customer loads.

Compared to IM Model:

- Models **voltage-driven recovery** of active/reactive power demand, not frequency-driven mechanical motion.
- Represents the slower phase of system response (seconds to minutes) vs. fast electromechanical transients.

Steady-state check: Setting $\dot{x}_p = \dot{x}_q = 0$ gives $P_d = P_0(V/V_0)^{N_{ps}}$, $Q_d = Q_0(V/V_0)^{N_{qs}}$, so ERL naturally reduces to a static voltage-dependent load, showing that when dynamics died out, the model reduces to a steady-state load.

ERL in TSC-OPF and Contrast with Induction Motor Model

TSC-OPF representation:

$$\dot{x} = f(x, y, p), \qquad 0 = g(x, y, p), \qquad h(x(t), y(t)) \leq 0.$$

- ERL contributes its states x_p, x_q to x, with recovery dynamics embedded in $f(x, y, p)$.
- $P_d(V, x_p), Q_d(V, x_q)$ enter the algebraic equations $g(x, y, p)$ for power balance at each bus.
- Voltage-recovery limits in $h(x(t), y(t))$ ensure $V_i(t) \geq V_{\text{min}}$ throughout the recovery window $[0, T_{\text{rec}}]$.

Contrast with IM model:

- **Induction Motor (IM):** physics-based; state $s = (\omega_s - \omega_r)/\omega_s$; captures fast electromechanical and frequency-coupled transients (0.1–2 s).
- **ERL:** empirical; states x_p, x_q; captures slow, voltage-driven recovery (1–60 s).

Why We Go Beyond Steady-State OPF

The bigger picture:

- Power systems are not static networks — they are **dynamic systems.**
- After every change — a fault, switching event, or sudden load or generation shift — voltages, currents, and frequencies evolve continuously before settling.
- Understanding and controlling these dynamics is essential for keeping the grid stable, secure, and resilient.

Four Building Blocks of System Dynamics

Four building blocks of system dynamics:

- **Transients:** capture the immediate electromagnetic wave response that propagates through the network.
- **Generator swing equations:** describe how machines adjust speed and angle to balance mechanical and electrical power.
- **Inverters:** the new generation interface that emulates inertia and voltage support with renewables.
- **Dynamic loads:** model how real demand recovers and interacts with voltage and frequency.

Why We Need Dynamic Optimization (TSC-OPF)

Why we need dynamic optimization (TSC-OPF):

- Steady-state OPF finds an economical operating point *only at equilibrium*.
- Transient Stability-Constrained OPF ensures that, even during those dynamic transitions, voltages remain safe, machines stay synchronized, and inverters and loads respond smoothly.